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The Ising model on the tetrahedron lattice III. Four-spin interactions

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Abstract. Our earlier study of the conventional nearest-neighbour Ising model on the tetrahedron lattice is extended to include four-spin interactions about each tetrahedron. The problem of deriving high-temperature susceptibility series can be converted into a form for which previous graphical techniques can be readily applied. Analysis of the series by Padé approximants and the ratio method suggests that the exponent γ has the constant value of 1·25 independent of the strength of the four-spin coupling, in agreement with universality.

The eight-vertex model (Baxter 1971, 1972) on a square lattice has been shown to be equivalent to two superimposed square Ising lattices coupled by a four-spin interaction (Kadanoff and Wegner 1971, Wu 1971). This has led to the study of multiple-spin interactions on various lattices: Ditzian (1972a, b), Wu (1972), Oitmaa and Gibberd (1973), Oitmaa (1974), Wood and Griffiths (1973, 1974), Griffiths and Wood (1973, 1974), Watts (1974), Jüngling and Obermair (1974). In quite a number of these models the critical exponents appear to vary continuously with a coupling parameter. It is not yet clear what features of such models lead to a departure from universality and it therefore seems worthwhile to carry out more model calculations. Recent studies on the high-temperature susceptibility series for the spin-$\frac{1}{2}$ nearest-neighbour Ising model on the tetrahedron lattice (Ho-Ting-Hun and Oitmaa 1975, to be referred to as I) have stimulated us to investigate further the validity of the universality hypothesis when the coupling of the four spins at the vertices of each tetrahedron is included in addition to the nearest-neighbour interactions. This model is, in a sense, a three-dimensional case of the model B investigated by Oitmaa and Gibberd (1973).

The model we consider can be described by the Hamiltonian

$$\mathscr{H} = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j - J_4\sum_{\langle klmn\rangle}\sigma_k\sigma_l\sigma_m\sigma_n - mH\sum_i\sigma_i \tag{1}$$

where $\langle ij\rangle$ denotes a pair of nearest-neighbour spins and $\langle klmn\rangle$ a quartet of spins at the vertices of a tetrahedron. No phase transition is possible in the case when only pure four-spin interactions, i.e. $J = 0$, are considered. The partition function for (1) is then

$$Z = \sum_{\{\sigma\}} e^{-\beta\mathscr{H}} = (\cosh K)^{3N}(\cosh K_4)^{N/2}\sum_{\{\sigma\}}\prod_{\langle ij\rangle}(1+v\sigma_i\sigma_j) \times \prod_{\langle klmn\rangle}(1+w\sigma_k\sigma_l\sigma_m\sigma_n)\prod_q \exp(h\sigma_q) \tag{2}$$

where $\beta = 1/kT$, $K = \beta J$, $K_4 = \beta J_4$, $v = \tanh K$, $w = \tanh K_4$, and $h = \beta mH$. Expanding the products on a particular tetrahedron, as in I, gives

$$Z = (\cosh K)^{3N}(\cosh K_4)^{N/2}[1+4v^3+3v^4+(3v^2+4v^3+v^6)w]^{N/2}$$
$$\times \sum_{\{\sigma\}} \prod_q \exp(h\sigma_q) \prod_{\text{tetrahedra}} [1+A(\sigma_1\sigma_2+\sigma_1\sigma_3+\sigma_1\sigma_4+\sigma_2\sigma_3+\sigma_2\sigma_4+\sigma_3\sigma_4)$$
$$+B\sigma_1\sigma_2\sigma_3\sigma_4] \tag{3}$$

with

$$A = \frac{(v+2v^2+2v^3+2v^4+v^5)(1+w)}{1+4v^3+3v^4+(3v^2+4v^3+v^6)w}$$
$$B = \frac{3v^2+4v^3+v^6+(1+4v^3+3v^4)w}{1+4v^3+3v^4+(3v^2+4v^3+v^6)w}. \tag{4}$$

The identity of the second product over the tetrahedra in (3) with equation (4) in I indicates that the graphical enumerations in I, namely counting dual graphs on the diamond lattice which is formed by joining the centres of the tetrahedra and adopting the same vertex weights, can be used in evaluating the zero-field susceptibility series from the expression

$$\chi = 1+2\sum_{\{\sigma\}} C_G A^m B^n. \tag{5}$$

Since on introducing the four-spin interactions the weight B has w as the lowest-order term, in order to obtain a 16-term series, some additional topological types of graphs must be included. These are shown in the appendix. The resulting series is

$$\chi = 1+6A+18A^2+54A^3+162A^4+486A^5+1446A^6+4194A^7+12234A^8$$
$$+35442A^9+102522A^{10}+294480A^{11}+847116A^{12}+2427840A^{13}$$
$$+6957600A^{14}+19878408A^{15}+56810148A^{16}+\ldots+B(12A^5$$
$$+72A^6+348A^7+1440A^8+5640A^9+20412A^{10}+71028A^{11}$$
$$+239844A^{12}+792480A^{13}+2552212A^{14}+8211900A^{15}+\ldots)$$
$$+B^2(6A^7+54A^8+414A^9+2106A^{10}+10038A^{11}+41506A^{12}$$
$$+165522A^{13}+619776A^{14}+\ldots)+B^3(72A^{10}+624A^{11}+3792A^{12}$$
$$+21192A^{13}+\ldots)+138A^{12}B^4+\ldots. \tag{6}$$

Using (4), the series can be written as

$$\chi = 1+6v+30v^2+138v^3+618v^4+2766v^5+12378v^6+55218v^7+245010v^8$$
$$+1081158v^9+4752054v^{10}+20842578v^{11}+91307598v^{12}$$
$$+399546882v^{13}+1745963826v^{14}+7618990770v^{15}$$
$$+33208413570v^{16}+\ldots+w(6v+48v^2+300v^3+1728v^4+9480v^5$$
$$+50280v^6+259272v^7+1304424v^8+6433548v^9+31271496v^{10}$$

$$
\begin{aligned}
&+150392112v^{11}+717040920v^{12}+3392059104v^{13}\\
&+15931856176v^{14}+74358739488v^{15}+\ldots)+w^2(18v^2+216v^3\\
&+1812v^4+13044v^5+85656v^6+525750v^7+3062262v^8\\
&+17153670v^9+93403772v^{10}+497680782v^{11}+2603885470v^{12}\\
&+13406343438v^{13}+68061286124v^{14}+\ldots)+w^3(54v^3+864v^4\\
&+9156v^5+79368v^6+602838v^7+4173252v^8+27079260v^9\\
&+167733120v^{10}+1001996004v^{11}+5806167840v^{12}\\
&+32777396232v^{13}+\ldots)+w^4(162v^4+3360v^5+42960v^6+427554v^7\\
&+3647112v^8+28127550v^9+201973050v^{10}+1371658944v^{11}\\
&+8896484202v^{12}+\ldots)+w^5(546v^5+13272v^6+191034v^7\\
&+2121924v^8+20157954v^9+171904872v^{10}+1349937978v^{11}+\ldots)\\
&+w^6(12v^5+1998v^6+51642v^7+821340v^8+10131078v^9\\
&+105987972v^{10}+\ldots)+w^7(72v^6+7404v^7+203436v^8\\
&+3541662v^9+\ldots)+w^8(390v^7+29214v^8+\ldots)+6w^9v^7+\ldots. \qquad (7)
\end{aligned}
$$

This series is by far the longest high-temperature series which has been obtained for a model with multiple-spin interactions. We note that putting $w=0$ recovers the series of I for the pure pair-interaction Ising model.

We have analysed the series by introducing a parameter $x = J_4/J$, expanding v and w in powers of K, and obtaining a series in K for a number of x values in the range $-2{\cdot}0 \leqslant x \leqslant 2{\cdot}5$. The resulting series has then been analysed by Padé approximants as described in I, namely by assuming at each x,

$$\chi(K,x) \sim C_0(x)\left(1-\frac{K}{K_c(x)}\right)^{-\gamma(x)}, \qquad K \to K_c(x)- \qquad (8)$$

and firstly estimating $K_c(x)$ from the logarithmic derivative series

$$\frac{\mathrm{d}}{\mathrm{d}K}\ln\chi(K,x) \simeq -\frac{\gamma(x)}{K-K_c(x)} \qquad (9)$$

with subsequent estimation of $\gamma(x)$ from Padé approximants to $(K-K_c(x))\,\mathrm{d}\ln\chi/\mathrm{d}K$ evaluated at $K = K_c(x)$. It is not possible to give rigorous error bounds in analysis of this kind. We have followed standard practice in making error estimates for $K_c(x)$ from the degree of scatter of the Padé table—these are confidence limits which are somewhat subjective. The uncertainty in $\gamma(x)$ depends on the uncertainty in $K_c(x)$ as well as on the scatter of the Padé table for a given $K_c(x)$. The estimated $\gamma(x)$ tend to be above $1{\cdot}25$ for $x<0$ and below $1{\cdot}25$ for $x>0$ as shown in figure 1. The slight increase in deviation of $\gamma(x)$ from $1{\cdot}25$ and the error limits as $|x|$ increases can probably be accounted for by the finiteness of the number of terms present in the series. It therefore appears that $\gamma(x) = 1{\cdot}25$ for all x and we therefore re-estimate $K_c(x)$ from Padé approximants to

$$(\chi(K,x))^{1/\gamma} \simeq \frac{K_c(x)C_0(x)^{1/\gamma}}{K_c(x)-K}. \qquad (10)$$

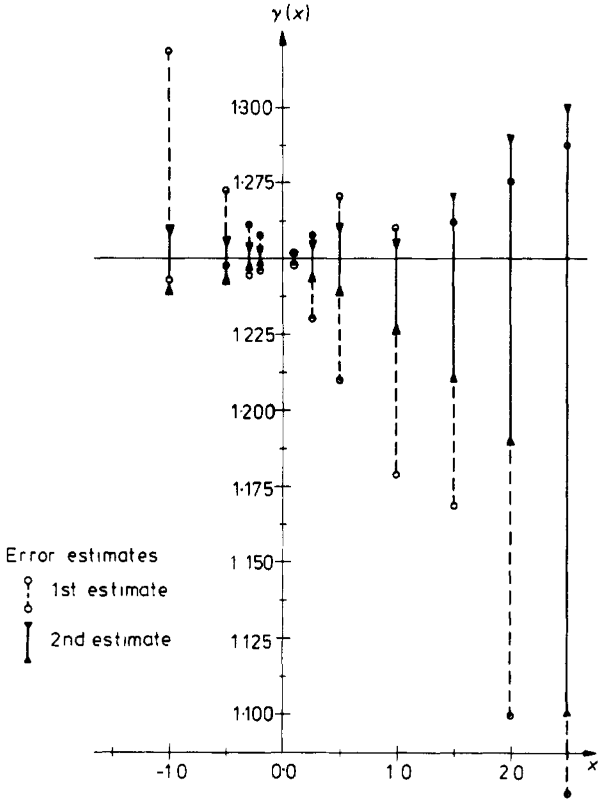

Figure 1. Plot of the estimated critical exponents $\gamma(x)$ with respect to the coupling strength parameter x.

Table 1 displays these estimates of $K_c(x)$ at some x and their convergence is quite reasonable even at large $|x|$. Using these more precise estimates of $K_c(x)$ we have re-estimated $\gamma(x)$ and these are also illustrated in figure 1, the reduced error limits of which reveal a better picture of $\gamma(x)$. This lends further support to the conjecture that $\gamma(x) = 1{\cdot}25$. Analysis using the ratio method yields similar results but with larger error limits. From all these calculations, it would then be plausible to conclude that $\gamma(x) = \gamma = 1{\cdot}25$ and is independent of the coupling strength parameter x.

We have also investigated the nature of the higher-order singularities, using the same approach as was used for the pure pair-interaction case by Oitmaa and Ho-Ting-Hun (1976). We assume an asymptotic form

$$\chi(K,x) = C_0(x)\tau^{-5/4} + C_1(x)\tau^{-\gamma_1(x)} + X_0(x) + C_2(x)\tau^{-\gamma_2(x)} + \dots \tag{11}$$

where $\tau = 1 - K/K_c(x)$.

The amplitudes $C_0(x)$, $C_1(x)$, $X_0(x)$ are all found to vary smoothly with x in the range $-0{\cdot}5 \leqslant x \leqslant 0{\cdot}5$. The analysis indicates that $\gamma_1(x) \simeq 0{\cdot}25$ and $\gamma_2(x) \simeq -0{\cdot}75$. It therefore appears that the presence of four-spin interactions in this model does not change the behaviour of either the leading or higher-order singularities but merely changes the amplitudes smoothly.

Table 1. Estimates of the critical point $K_c(x)$ of the tetrahedron lattice obtained from Padé approximants to $(\chi(K, x))^{4/5}$ at particular x values.

$[N, D]$ \ x	−2·0	−0·5	0·5	1·0	2·0	2·6
5, 11	0·457	0·2594	0·2212	0·2086	0·1905	0·1825
6, 10	0·449	0·2593	0·2212	0·2088	0·1896	0·1819
7, 9	0·449	0·2593	0·2212	0·2087	0·1906	0·1826
8, 8	0·449	0·2593	0·2212	0·2086	0·1906	0·1826
9, 7	0·451	0·2593	0·2212	0·2087	0·1906	0·1826
10, 6	0·449	0·2593	0·2212	0·2087	0·1906	0·1826
5, 10	0·455	0·2593	0·2212	0·2086	0·1904	0·1822
6, 9	0·450	0·2493	0·2211	0·2086	0·1904	0·1822
7, 8	0·449	0·2593	0·2211	0·2088	0·1906	0·1827
8, 7	0·449	0·2593	0·2212	0·2088	0·1906	0·1827
9, 6	0·449	0·2593	0·2212	0·2087	0·1906	0·1827

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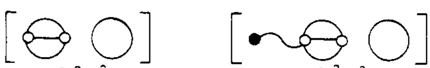
Appendix

Because the vertex weight B is of order v (rather than v^2 as in I) a number of additional topological types of graphs contribute up to order v^{16}. These are listed below, together with their weights. As in I, n denotes the number of bonds in a graph and second-order vertices of type —●— are suppressed.

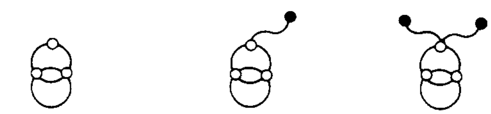
A.1. Graphs proportional to A^rB^2 ($r \leqslant 14$)

$-(n+1)A^{n-3}B^2$ $6A^{n-3}B^2$ $9A^{n-3}B^2$

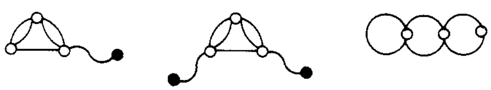


$A^{n-3}B^2$ $3A^{n-3}B^2$

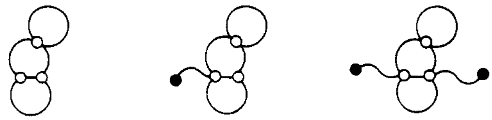
A.2. Graphs proportional to A^rB^3 ($r \leqslant 13$)



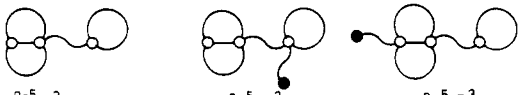
$(n-4)A^{n-5}B^3$ $3A^{n-5}B^3$ $9A^{n-5}B^3$



$3A^{n-5}B^3$ $9A^{n-5}B^3$ $(n-4)A^{n-5}B^3$

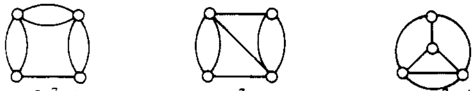


$A^{n-5}B^3$ $\quad$ $3A^{n-5}B^3$ $\quad$ $9A^{n-5}B^3$



$A^{n-5}B^3$ $\quad$ $3A^{n-5}B^3$ $\quad$ $3A^{n-5}B^3$

A.3. Graphs proportional to A^rB^4 $(r \leqslant 12)$



$A^{n-7}B^4$ $\quad$ $A^{n-7}B^4$ $\quad$ $A^{n-7}B^4$

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